and thus it must be true of all subsequent pairs. The same must also be true in the other direction from P. We conclude that the line we are considering must contain a set of intervals of length d with the periodicity of the 2-intersections, inside of which no 1-intersection can lie. This, however, is impossible because the period of the 1-intersections is incommensurate with the period of the 2-intersections. Thus d must be zero, the 2-intersections must indeed coincide, and the position of family 2 is fixed with respect to families 0 and 1.

The rest of the proof is simple: any other family 3 that is not parallel to family 0 must have a spacing between intersections along a line in family 0 that is incommensurate with the spacings of either family 1 or of family 2 on family 0 (since if it were commensurate with both then families 1 and 2 would have commensurate spacings). Therefore, by repeating the first part of the argument we can conclude that the position of family 3 is fixed with respect to either family 1 or family 2. In this way the positions of all families are fixed except for those given by families parallel to family 0. But these can now be fixed, in the same way, with respect to families not parallel to 0. Thus the grids are indeed identical except for a possible translation.

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Crystallography, Geometry and Physics in Higher Dimensions. VI. Geometrical 'WPV' Symbols for the 371 Crystallographic Mono-Incommensurate Space Groups in Four-Dimensional Space

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Abstract

A geometrical 'WPV' notation for the 371 crystallographic space groups describing mono-incommensurate phases of physical space in four-dimensional space is proposed, which completes the geometrical 'WPV' notation for all crystallographic point symmetry groups. The WPV symbols are given for the 76 mono-incommensurate arithmetic classes, or Z classes. Definitions and some examples of Z classes, Bravais types, Bravais flocks, Q classes (or geometrical classes or point groups), holohedries and crystal families both in the physical space and the superspace $\mathbb{E}^4$ are given.

Introduction

In a previous paper (Weigel, Phan & Veysseyre, 1987) we have given a simple geometric symbol, the 'WPV' symbol, for each of the 227 crystallographic point symmetry groups (PSGs) of the four-dimensional space $\mathbb{E}^4$. In this article we propose a WPV symbol for 371 crystallographic symmetry space groups (SSGs) belonging to the seven crystal systems of $\mathbb{E}^4$ describing the mono-incommensurate phases of the physical space.

A symmetry space group of the Euclidean space $\mathbb{R}^n$ is the group of all the crystallographic symmetry operations (SOs), or isometries, mapping one crystal structure onto itself. A space group is always an infinite group because a crystal structure has infinitely many symmetry translations. The set of all the translation vectors of $\mathbb{E}^n$ mapping a crystal structure onto itself is the lattice of this structure. A lattice of $\mathbb{E}^n$ is defined by n linearly independent vectors $\mathbf{e}_i$ (i varying from 1 to n). So it depends, in the most general case, on n parameters of length and $n(n-1)/2$ parameters of angle.

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Table 1. *Monoclinic system*

The two subtables (*a*) and (*b*) give two possible classifications of the symmetry space groups of the monoclinic system of the physical space $\mathbb{E}^3$. (*a*) gives the WPV symbols of the three *Q* classes (first column), the six *Z* classes (second column) and the 13 space-group types (third column). (*b*) gives the classification of the space-group types into the two Bravais flocks of this system, *i.e.* monoclinic *P* and monoclinic *B*.

(*a*)

Q classes or point groups	*Z* classes	Space-group types
2	$2P$	$P2$; $P2_1$
	$2B$	$B2$
m	mP	Pm; Pb
	mB	Bm; Bb
$2/m$	$2/mP$	$P2/m$; $P2_1/m$; $P2/b$; $P2_1/b$
	$2/mB$	$B2/m$; $B2/b$

(*b*)

Bravais types of lattices	Bravais flocks
Monoclinic *P*	$P2$; $P2_1$; Pm; Pb $P2/m$; $P2_1/m$; $P2/b$; $P2_1/b$
Monoclinic *B*	$B2$; Bm; Bb; $B2/m$; $B2/b$

In addition to the translations, another type of SO belongs to the SSG of a symmorphic crystal structure (see definition later); these are the point symmetry operations (PSOs) that we have defined and classified in a previous paper (Weigel, Veysseyre, Phan, Effantin & Billiet, 1984). These PSOs are called 'linear constituents' by Brown, Bülow, Neubüser, Wondratschek & Zassenhaus (1978) and they make up a finite group isomorphic to the factor group of the SSG by the normal subgroup of all symmetry translations. This finite group is the PSG of the cell. If we choose a lattice basis, these PSOs are described by unimodular matrices of order *n* (their entries are integers and their determinant equal to ± 1).

As the number of SSGs in $\mathbb{E}^n$ is infinite it was necessary to classify them into a finite number of classes. Different classifications have been proposed (Brown *et al.*, 1978; Schwarzenberger, 1980; Neubüser, Plesken & Wondratschek, 1981). Here we illustrate the classification given by Brown *et al.* (1978) with some examples in physical space in order to prepare the way for § I of this paper.

The basic notion is the one of *space-group type.* For instance, let us consider the triclinic lattice of $\mathbb{E}^3$ and the space-group type $P\bar{1}$. This lattice is defined by three parameters of length (a, b, c) and three of angles (α, β, γ).

The space-group type of the triclinic lattice contains all the SSGs of the infinity of triclinic lattices, defined by all the possible values of the six parameters (a, b, c, α, β, γ). In mathematical terms, we say that these SSGs are equivalent modulo an affine mapping. So there are 17 space-group types in $\mathbb{E}^2$, 219 in $\mathbb{E}^3$ and 4783 in $\mathbb{E}^4$; or, rather 230 in $\mathbb{E}^3$ and 4895 in $\mathbb{E}^4$ because 11 space-group types in $\mathbb{E}^3$ and 112 in $\mathbb{E}^4$ split into enantiomorphic pairs.

A SSG is said to be *symmorphic* if its PSG is one of its subgroups; for example the monoclinic SSG $P2$ is symmorphic, but the monoclinic SSG $P2_1$ is not symmorphic.

Now let us consider a definite crystal structure. Different lattice bases can be chosen to describe the SOs of this structure. All the SSGs so obtained belong to an 'arithmetic class' or *Z class.* For instance in $\mathbb{E}^3$ the monoclinic *Z* class $2B$ contains one space-group type $B2$ while the monoclinic $2P$ contains two space-group types, $P2$ and $P2_1$ (see Table 1). A *Z* class is therefore associated with a structure and a PSG. This PSG maps the crystal structure onto itself but the associated lattice may have other isometries: let us consider the SSG $B2$; the associated (empty) lattice has for SSG $B2/m$. In this way, we define a particular *Z* class or Bravais *Z* class which contains the SSGs which describe all the isometries of an (empty) lattice with respect to all possible lattice bases. So the monoclinic crystal system of $\mathbb{E}^3$ has two Bravais *Z* classes, $2/mP$ and $2/mB$ (Table 1*a*).

Two lattices belong to the same Bravais type of lattice if they both determine the same Bravais *Z* class. The concept of centring lattice is the same as for the two- or three-dimensional space; it is always defined relative to a basis. 'A lattice *L* in $\mathbb{E}^n$ is considered centred with respect to a basis *B* if all vectors in $\mathbb{E}^n$ that have integral coefficients with respect to *B* belong to *L*, but if there exists some vector of *L* whose coefficients with respect to *B* are not all integral' (Brown *et al.*, 1978). So a Bravais type of lattice can be primitive or centred, *e.g.* monoclinic *B*. There are 14 Bravais types of lattices in $\mathbb{E}^3$ and 64 in $\mathbb{E}^4$.

Mathematically, it may be proved that each *Z* class *A* may be associated with a particular and unique Bravais *Z* class *B* in the following sense: each finite unimodular (f.u. for short) group of *A* is a subgroup of some f.u. group in *B* but it is not a subgroup of any f.u. group belonging to a Bravais *Z* class of smaller order.

The set of *Z* classes associated with a particular Bravais *Z* class is called a *Bravais flock.* So there are 14 Bravais flocks and 73 *Z* classes in $\mathbb{E}^3$ and 64 and 710, respectively, in $\mathbb{E}^4$. We can give an example in $\mathbb{E}^3$: the Bravais type of lattice monoclinic *B* is associated with the Bravais flock ($B2$, Bm, Bb, $B2/m$, $B2/b$) (see Table 1*b*).

If we just consider the PSG of the space-group type, another classification may be obtained – the *geometric classes* or *Q* classes or *point groups* – which describe the macroscopic properties of a crystal structure. Table 1 lists the *Q* classes of the monoclinic system.

In this case, it is possible to use a general basis and not necessary a lattice basis. The *Q* classes or point groups realize a classification of the *Z* classes. Two SSGs belong to the same *Q* class if their PSGs are equivalent modulo a real non-singular matrix. If among the Z classes belonging to a Q class there is at least one Bravais Z class, this Q class is said to be a *holohedry*.

There are 32 Q classes or crystallographic point groups and 7 holohedries in $\mathbb{E}^3$ and 227 and 33, respectively, in $\mathbb{E}^4$.

In $\mathbb{E}^3$ the Q class $2/m$ is a holohedry because $2/m\,P$ (and also $2/m\,B$) is a Bravais Z class.

The Q classes and the Bravais flocks are two subdivisions of the set of all the Z classes and of all the types of SSGs. The definition of *crystal families* is a dimension-independent classification whereas the various definitions of a *crystal system* depend on the dimension of the space.

'A crystal family is the smallest set of space-group types containing, for any of its members, all space-group types of the Bravais flock and all space-group types of the Q class, to which this member belongs' (Neubüser *et al.*, 1981); *e.g.* in $\mathbb{E}^3$ there are six crystal families: triclinic, monoclinic, orthorhombic, tetragonal, hexagonal and cubic, and there are 23 crystal families in $\mathbb{E}^4$ (Brown *et al.*, 1978; Weigel *et al.*, 1987).

Into the spaces of 2, 3 or 4 dimensions, it is possible to associate with a Q class a unique holohedry in the same way as a Z class is associated with a unique Bravais Z class. Then a crystal system is the set of the Q classes associated with the same holohedry. So there are seven crystal systems in $\mathbb{E}^3$, because the hexagonal family splits into two crystal systems: the rhombohedral system and the hexagonal system. There are 33 crystal systems in $\mathbb{E}^4$. But this definition is not exact for higher-dimensional spaces (Neubüser *et al.*, 1981), in which it is not convenient to use the concept of crystal system.

We summarize all these notions with the monoclinic system of $\mathbb{E}^3$; the cell of this crystal structure is a right hyperprism based on a parallelogram.

This crystal system contains three Q classes or point groups: (1) the holohedry $2/m$ of order 4 (the four elements are $1, 2_{XY}, m_Z, \bar{1}_{XYZ}$); (2) the Q class m; (3) the Q class 2. m and 2 are subgroups of the holohedry $2/m$.

The monoclinic system may be primitive P or with one centred face: the (a, c) face. This type of centring is denoted B in *International Tables for Crystallography* (1987) (*IT*). So this system contains two Bravais types of lattice, monoclinic P and monoclinic B. The Q class $2/m$ contains two Z classes: $2/m\,P$ and $2/m\,B$.

The Z class $2/m\,P$ contains four space-group types: $P2/m$ which is symmorphic, $P2_1/m$, $P2/b$ and $P2_1/b$; the Z class $2/m\,B$ contains only two space-group types, $B2/m$, which is symmorphic, and $B2/b$. The other space-group types are given in Table 1(a). The classification into Bravais types of lattice is given in Table 1(b).

I. WPV symbols for the SSGs of the seven mono-incommensurate crystal systems of $\mathbb{E}^4$

We recall that a crystal structure is said to be mono-incommensurate (MI for short) if the experimental diffraction pattern of this phase is described by four Miller indices:

$$H = h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^* + m\mathbf{q}^* \qquad (1)$$

where h, k, l, m are integers,

$$\mathbf{q}^* = \alpha\mathbf{a}^* + \beta\mathbf{b}^* + \gamma\mathbf{c}^* \qquad (2)$$

and at least one of the three entries is irrational.

de Wolff (1974) has proved that the matrices which describe the PSOs of such a phase have the following form with respect to an orthonormal basis:

$$\left(\begin{array}{c|cc} Q & \multicolumn{2}{c}{0} \\ \hline 0 & \begin{array}{cc} \varepsilon & 0 \\ 0 & \varepsilon \end{array} \end{array}\right)$$

where Q is a 2×2 matrix and $\varepsilon = \pm1$. There are six types of $(\mathrm{MI})^+$ PSO if $\varepsilon = +1$ and five types of $(\mathrm{MI})^-$ PSO if $\varepsilon = -1$ (Weigel & Bertaut, 1986; Veysseyre & Weigel, 1989). The $(\mathrm{MI})^+$ PSOs have for WPV symbols:

$$1, 2_{XY}, 3^{\pm1}_{XY}, 4^{\pm1}_{XY}, 6^{\pm1}_{XY}, m_x$$

and the $(\mathrm{MI})^-$ PSOs:

$$2_{ZT}, \bar{1}_4, 2_{ZT}3^{\pm1}_{XY}, 2_{ZT}6^{\pm1}_{XY}, \bar{1}_{XZT}.$$

In formula (2), if the three coefficients are irrational, only the PSOs 1 and $\bar{1}_4$ may appear. If two coefficients are irrational, only the PSOs 1, $\bar{1}_4$, m_X, $\bar{1}_{XZT}$ may appear. Lastly, if just one coefficient is irrational, all the (MI) PSOs may appear. A PSG is a (MI) PSG if all its elements are (MI) PSOs.

A crystal system is a (MI) crystal system if it contains only (MI) PSGs.

Just seven crystal systems in the four-dimensional space $\mathbb{E}^4$ can describe the (MI) crystal structure (de Wolff, 1974; Veysseyre & Weigel, 1989). They correspond to the systems 1, 2, 3, 4, 7, 8 and 9 (Wondratschek, Bülow & Neubüser, 1971).

In Table 2 we give the geometrical names of these seven systems, the WPV symbols of the corresponding holohedries and the Bravais types of each system. There are 16 Bravais types. These seven systems contain 30 Q classes, *i.e.* 30 PSGs, 76 Z classes and 371 space groups. Brown *et al.* (1978) have given a listing of all the 4783 space-group types of $\mathbb{E}^4$. For each SSG, some SOs, generators of the group, are given.

They are described by 5×5 matrices with respect to a chosen basis lattice. Any matrix of these SOs is

Table 2. *The* 16 *Bravais types of the seven mono-incommensurate systems*

The numerotations of the crystal family and of the crystal system are recalled in the first and the second columns. We give the geometrical name of each mono-incommensurate system in the third column and the WPV symbol of its holohedry in the fourth column. Finally in the fifth column the 16 Bravais types are listed with respect to the corresponding systems with the following centring types (the coordinates of all the centring points are explicitly written).

Primitive:	P	
One centred face:	$S(Z, T)$	$(0, 0, 0, 0)$ $(0, 0, \frac{1}{2}, \frac{1}{2})$
	$S(Y, Z)$	$(0, 0, 0, 0)$ $(0, \frac{1}{2}, \frac{1}{2}, 0)$
Two centred faces:	$D(X, T)(Y, Z)$	$(0, 0, 0, 0)$ $(\frac{1}{2}, 0, 0, \frac{1}{2})$ $(0, \frac{1}{2}, \frac{1}{2}, 0)$ $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
Three centred faces:	$F(Y, Z, T)$	$(0, 0, 0, 0)$ $(0, 0, \frac{1}{2}, \frac{1}{2})$ $(0, \frac{1}{2}, 0, \frac{1}{2})$ $(0, \frac{1}{2}, \frac{1}{2}, 0)$
Body centred:	$I(Y, Z, T)$	$(0, 0, 0, 0)$ $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
Rhombohedron:	$R(Y, Z, T)$	$(0, 0, 0, 0)$ $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$

Family	System	Geometrical name	'WPV' symbol of the holohedry	Bravais types
I	01	Hexaclinic	$\bar{1}_4$	P
II	02	Right hyperprism based on parallelepiped (XYZ)	$\bar{1}\perp m$	P, $S(Z, T)$
III	03	Di orthogonal parallelograms (XY) (ZT)	$2\perp 2$	P, $S(Y, Z)$, $D(X, T)(Y, Z)$
IV	04	Orthogonal parallelogram (XY) rectangle (ZT)	$2\perp 2, m, m$	P, $S(Y, Z)$, $S(Z, T)$, $I(Y, Z, T)$, $D(X, T)(Y, Z)$, $F(Y, Z, T)$
VI	07	Orthogonal parallelogram (XY) square (ZT)	$2\perp 4, m, m$	P, $I(Y, Z, T)$
VII	08	Orthogonal parallelogram (XY) hexagon (ZT) $R(Y, Z, T)$ centred	$26, m, \bar{1}$	P, $R(Y, Z, T)$*
	09	Orthogonal parallelogram (XY) hexagon (ZT)	$2\perp 6, m, m$	

* Only for the system 08.

determined by

$$\left(\begin{array}{cccc|c} & & & & t_1 \\ & A & & & t_2 \\ & & & & t_3 \\ & & & & t_4 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad t=\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$$

where A is a 4×4 matrix of a PSO (Weigel *et al.*, 1984) and t is a rational 4-column or 'translation vector'.

The symmorphic space group of each Z class which is the head of the Z class corresponds to a SSG with $t=0$ for each generator.

It is clear that other generators would be given in many cases. The matrices A are generators of the corresponding PSG. These generators appear often in the WPV notation of the PSG (Weigel *et al.*, 1987).

Other generators are sometimes used and in our notation we frequently indicate more generators than necessary; the same is true for Hermann–Mauguin notation in physical space.

Let us give one example. In the system VI.07 or orthogonal parallelogram (XY) square (ZT) the PSG 06 is described by two generators $4_{ZT}^{\pm 1}$ and m_Z; its order is 8. Respecting the notation of Hermann–Mauguin, we have called it 4, *m*, *m*. Thus its elements are

$$1, 4_{ZT}^{\pm 1}, 2_{ZT}, m_Z, m_T, m_{Z+T}, m_{Z-T}.$$

Other generators, such as m_T and m_{Z+T}, for instance, are possible.

We have previously given the matrix of a general SO. In the physical space $\mathbb{E}^3$, two types of glide SOs exist: the helirotations, for example 4_1 (or $4_{C/4}$) and the glide reflections as **a** (or $m_{\mathbf{a}/2}$ or $m_{\mathbf{a}}$ for short) and n (or $m_{\mathbf{a}+\mathbf{b}/2}$ or $m_{\mathbf{a}+\mathbf{b}}$). The corresponding matrices are the matrices 1, 2 and 3. In the space $\mathbb{E}^4$, three types appear:

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 1. Helirotation 4_1.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 2. Glide reflection $a = m_{\mathbf{a}}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 3. Glide reflection $n = m_{\mathbf{a}+\mathbf{b}}$.

Glide inversions: $\bar{1}_{\mathbf{a}}$, for example, is the product of the inversion $\bar{1}_{YZT}$ and the translation of vectors $\mathbf{a}/2$ parallel to the x axis (matrix 4). We recall that **a**, **b**, **c** and **d** are the vectors which define the crystal cell whereas (X, Y, Z, T) denote the mathematical basis.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 4. Glide inversion $\bar{1}_{\mathbf{a}}$.

Glide rotations: $2_{\mathbf{a}+\mathbf{b}}$ is the product of the rotation 2_{ZT} and the translation of vector $(\mathbf{a}+\mathbf{b})/2$. $4^1_{\mathbf{a}(ZT)}$ is the product of the rotation 4^1_{ZT} and the translation of vector $\mathbf{a}/2$ (matrix 5).

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 5. Glide rotation $4^1_{\mathbf{a}(ZT)}$.

Glide reflections: $m_{\mathbf{a}+\mathbf{b}+\mathbf{c}}$, for example, is the product of the reflection m_T about the hyperplane (XYZ) and the translation of vector $(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$.

When the vector of the translation is $\mathbf{a}/2$ or $(\mathbf{a}+\mathbf{b})/2$ the translation is denoted $\mathbf{a}$ or $\mathbf{a}+\mathbf{b}$, for short, as usual in $\mathbb{E}^3$. In the other cases, *e.g.* if the glide vector is $\mathbf{a}/4$ or $(\mathbf{b}+\mathbf{c})/3$ the complete vector is kept in the symbol, as for the glide rotation $3_{\mathbf{b}/3(ZT)}$ (matrix 6).

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 6. Glide rotation $3_{\mathbf{b}/3(ZT)}$.

Now we are going to describe one Q class, VI-07-06, the sixth Q class of the seventh system of the sixth family.

This Q class (see Table 3) has two Z classes, VI-07-06-01 and VI-07-06-02 (notation of Brown *et al.*, 1978). The first Z class corresponds to the Bravais type P primitive. It is denoted 4, *m*, *m P*. It includes ten space groups. The first is the symmorphic space group denoted $P4, m, m$. The second is generated by $4^1_{\mathbf{a}}$ and m_T (matrices 5 and 7). The leading elements

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 7. Reflection m_T.

of this space group are*

$$1,\ 2_{ZT},\ m_Z,\ m_T,\ m_{\mathbf{a}(Z+T)},\ m_{\mathbf{a}(Z-T)},\ 4^{+1}_{\mathbf{a}(ZT)},\ 4^{-1}_{\mathbf{a}(ZT)}.$$

We denote it by $P\,4_{\mathbf{a}}, m, m$. The other space groups of this Z class are described in the same way.

The second Z class, VI-07-06-02, corresponds to the Bravais type I (Y, Z, T) centred. It is denoted by 4, *m*, *m I*. It includes nine space groups. The first is the symmorphic space group $I\ (Y, Z, T)\ 4, m, m$. The third is generated by $4^1_{\mathbf{b}/4}$ and $m_{\mathbf{b}+\mathbf{c}}$ (matrices 8 and 9). Its leading elements are:

$$1,\ 2_{\mathbf{b}(ZT)},\ m_Z,\ m_{\mathbf{b}+\mathbf{c}(T)},\ m_{\mathbf{b}/4(Z+T)},$$
$$m_{\mathbf{b}/4(Z-T)},\ 4^{+1}_{\mathbf{b}/4(ZT)},\ 4^{-1}_{\mathbf{b}/4(ZT)}.$$

We write it as $I\ (Y, Z, T)\ 4_{\mathbf{b}/4}, m, m_{\mathbf{b}+\mathbf{c}}$.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 8. Glide rotation $4^1_{\mathbf{b}/4(ZT)}$.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Matrix number 9. Glide reflection $m_{\mathbf{b}+\mathbf{c}(T)}$.

II. Table of the 371 mono-incommensurate space groups with their WPV symbols

The 371 space groups of the seven mono-incommensurate systems are listed in Table 3. They are classified by systems, within each system by Q classes and within each Q class by Z classes. We recall the notation of Brown *et al.* (1978) and we give our 'WPV' notation for each space-group type. So 04-01-02-003 or $S(Z, T)\ 2, m_{\mathbf{b}}, m_{\mathbf{b}}$, in WPV notation, is the third space group of the second Z class $2, m, m\ S(Z, T)$ associated with the first point group $2, m, m$ of the system 04: orthogonal parallelogram (XY) rectangle (ZT). For each Q class or point group some generators are given; these generators appear in the WPV notation. They are often too many as explained previously. For example we have given three generators for the Q class 04-01: 2_{ZT}, m_Z, m_T, but only two of them would be sufficient.

Sometimes the disposition of the geometrical supports of generators with respect to the cell of the crystal does split some Q classes into two parts. This

* We recall that a SSG is an infinite group. All its elements are obtained by products of these leading elements and of all the translations of the lattice.

Table 3. *WPV notation of the* 371 *crystallographic SSGs of the seven mono-incommensurate systems*

This table is divided into subtables, one for each system. Each subtable is divided into several parts, one for each Q class belonging to the corresponding system. For each Q-class heading generator SOs are written. The space groups are regrouped into Z classes characterized by their centring types.

The notation is as follows:

System-Q class-Z class-Space group.

For instance, 09-06-02-001 is the first space group of the second Z class of the sixth Q class of the ninth system.

I-1 Hexaclinic

01-01-01-001	P	1	01-02-01-001	P	$\bar{1}_4$

II-2 Right hyperprism based on parallelepiped (XYZ)

		m_T
02-01-01-001	P	m
002		m_c
02-001	$S(Z, T)$	m
002		m_a

		$\bar{1}_{XYZ}$
02-02-01-001	P	$\bar{1}$
002		$\bar{1}_d$
02-001	$S(Z, T)$	$\bar{1}$

		$\bar{1}_{XYZ}$,		m_T
02-03-01-001	P	$\bar{1}$	$\perp$	m
002		$\bar{1}_d$	$\perp$	m
003		$\bar{1}$	$\perp$	m_c
004		$\bar{1}_d$	$\perp$	m_c
02-001	$S(Z, T)$	$\bar{1}$	$\perp$	m
002		$\bar{1}$	$\perp$	m_b

III-3 Di orthogonal parallelograms (XY) (ZT)

		2_{XY}
03-01-01-001	P	2
002		2_d
02-001	$S(Y, Z)$	2
002		2_d
03-001	$D(X, T)(Y, Z)$	2

		2_{XY},		2_{ZT}
03-02-01-001	P	2	$\perp$	2
002		2_d	$\perp$	2
003		2_d	$\perp$	2_b
02-001	$S(Y, Z)$	2	$\perp$	2
002		2_d	$\perp$	2
003		2_d	$\perp$	2_a
03-001	$D(X, T)(Y, Z)$	2	$\perp$	2

fact already happens in the physical space $\mathbb{E}^3$: $3m1\ P$ and $31m\ P$ are two different Z classes of the Q class $3m$. In the system 07 of $\mathbb{E}^4$, orthogonal parallelogram (XY) square (ZT), the Q class 07-04 or $24, m, \bar{1}$ is divided into two subdivisions $24, \bar{1}, m$ and $24, m, \bar{1}$. Within the first part the two Z classes $24, \bar{1}, m\ P$ and $24, \bar{1}, m\ I$ are generated by 2_{XY}, 4^1_{ZT}, $\bar{1}_{XYZ}$, and m_{Z-T}. The leading elements of the symmorphic space group $P\ 24, \bar{1}, m$ are

$$1, 2_{XY}, 4^{\pm 1}_{ZT}, 2_{ZT}, \bar{1}_{XYZ}, \bar{1}_{XYT}, m_{Z-T}, m_{Z+T}.$$

Within the second part the Z classes $24, m, \bar{1}\ P$ and $24, m, \bar{1}\ I$ are generated by $2_{XY}, 4^1_{ZT}, m_Z$, and $\bar{1}_{XYZ-T}$; the leading elements of the symmorphic space group $P\ 24, m, \bar{1}$ are

$$1, 2_{XY} 4^{\pm 1}_{ZT}, 2_{ZT}, m_Z, m_T, \bar{1}_{XYZ-T}, \bar{1}_{XYZ+T}.$$

Another classification of the space-group types, different from the previous one, is possible. Instead of regrouping the SSGs into Z classes we gather them into Bravais flocks, *i.e.* according to the types of lattices.

Let us consider for instance the system 03 or di orthogonal parallelograms; it contains three Bravais types of lattices: di orthogonal parallelograms P; di orthogonal parallelograms $S(Y, Z)$; di orthogonal parallelograms $D(X, T)(Y, Z)$. These three types of centring are explained in the caption to Table 2.

The Bravais Z class of the first type is the Z class $2\perp 2P$. In addition it contains the Z class $2P$; the first one, $2\perp 2P$, contains three SSGs, $P2\perp 2$, $P2_d\perp 2$, $P2_d\perp 2_b$; the second one, $2P$, contains two SSGs, $P2$ and $P2_d$; and so on for the other two Bravais types.

We summarize this classification for all the SSGs of the system 03 in Table 4.

In Tables 2–4 different types of centring appear. We are going to detail three examples which present some particularities.

First, let us consider the di orthogonal parallelograms $(XY)(ZT)$ family (or family III) and its holohedry $2\perp 2$ which defines the Q class 03-02. This Q class contains three Z classes:

$$2\perp 2P;\quad 2\perp 2S(Y, Z);\quad 2\perp 2D(X, T)(Y, Z).$$

Table 3 (*cont.* 1)

IV-4 Orthogonal parallelogram (XY) rectangle (ZT)

		2_{ZT},	m_Z,	m_T
04-01-01-001	P	2,	m,	m
002		2,	m_d,	m
003		2_b,	m_b,	m
004		2_b,	m_{b+d},	m
005		2,	m_d,	m_c
006		2_b,	m_b,	m_c
007		2_b,	m_{b+d},	m_c
008		2,	m_b,	m_b
009		2,	m_{b+d},	m_b
010		2_{a+b},	m_a,	m_b
011		2_{a+b},	m_{a+d},	m_b
012		2,	m_{b+d},	m_{b+c}
013		2_{a+b},	m_{a+d},	m_{b+c}
02-001	$S(Z, T)$	2,	m,	m
002		2_b,	m_b,	m
003		2,	m_b,	m_b
004		2_{a+b},	m_a,	m_b
03-001	$S(Y, Z)$	2,	m,	m
002		2_a,	m,	m_a
003		2,	m_d,	m
004		2_a,	m_d,	m_a
005		2_a,	m_a,	m
006		2,	m_a,	m_a
007		2_a,	m_{a+d},	m
008		2,	m_{a+d};	m_a
009		2_b,	m,	m_b
010		2_b,	m_d,	m_b
011		2_{a+b},	m_a,	m_b
012		2_{a+b},	m_{a+d},	m_b
04-001	$I(Y, Z, T)$	2,	m,	m
002		2_a,	m,	m_a
003		2,	m_a,	m_a
004		2_b,	m_b,	m
005		2_{a+b},	m_b,	m_a
006		2_b,	m_{a+b},	m_a
007		2_b,	m_b,	m_c
05-001	$D(X, T)(Y, Z)$	2,	m,	m
002		2_a,	m_a,	m
003		2_{a+b},	m_a,	m_b
06-001	$F(Y, Z, T)$	2,	m,	m
002		2,	$m_{(b+d)/4}$,	$m_{(b+c)/4}$
003		2_a,	m_a,	m
004		2_a,	$m_{(2a+b+d)/4}$,	$m_{(b+c)/4}$
005		2,	m_a,	m_a

		2_{XY},		m_Z,	$\bar{1}_{XYZ}$
04-02-01-001	P	2	/	m,	$\bar{1}$
002		2_c	/	m,	$\bar{1}$
003		2_d	/	m,	$\bar{1}_d$
004		2_{c+d}	/	m,	$\bar{1}_d$
005		2_d	/	m_d,	$\bar{1}$
006		2_{c+d}	/	m_d,	$\bar{1}$
007		2	/	m_d,	$\bar{1}_d$
008		2_c	/	m_d,	$\bar{1}_d$
009		2	/	m_b,	$\bar{1}$
010		2_c	/	m_b,	$\bar{1}$
011		2_d	/	m_b,	$\bar{1}_d$
012		2_{c+d}	/	m_b,	$\bar{1}_d$
013		2_d	/	m_{b+d},	$\bar{1}$
014		2_{c+d}	/	m_{b+d},	$\bar{1}$
015		2	/	m_{b+d},	$\bar{1}_d$
016		2_c	/	m_{b+d},	$\bar{1}_d$
02-001	$S(Z, T)$	2	/	m,	$\bar{1}$
002		2_c	/	m,	$\bar{1}$
003		2	/	m_b,	$\bar{1}$
004		2_c	/	m_b,	$\bar{1}$
03-001	$S(Y, Z)$	2	/	m,	$\bar{1}$
002		2	/	m_a,	$\bar{1}$
003		2_d	/	m,	$\bar{1}_d$
004		2_d	/	m_a,	$\bar{1}_d$
005		2_d	/	m_d,	$\bar{1}$
006		2_d	/	m_{a+d},	$\bar{1}$
007		2	/	m_d,	$\bar{1}_d$
008		2	/	m_{a+d},	$\bar{1}_d$
04-001	$S(Y, T)$	2	/	m,	$\bar{1}$
002		2_c	/	m,	$\bar{1}$
003		2	/	m_a,	$\bar{1}$
004		2_c	/	m_a,	$\bar{1}$
005		2	/	m_b,	$\bar{1}$
006		2_c	/	m_b,	$\bar{1}$
05-001	$I(Y, Z, T)$	2	/	m,	$\bar{1}$
002		2_c	/	m,	$\bar{1}$
003		2	/	m_a,	$\bar{1}$
004		2_c	/	m_a,	$\bar{1}$
005		2	/	m_b,	$\bar{1}$
006		2_c	/	m_b,	$\bar{1}$
06-001	$D(X, T)(Y, Z)$	2	/	m,	$\bar{1}$
002		2	/	m_a,	$\bar{1}$
07-001	$F(Y, Z, T)$	2	/	m,	$\bar{1}$
002		2	/	m_a,	$\bar{1}$
003		$2_{(c-d)/4}$	/	$m_{(b+d)/4}$,	$\bar{1}$

		2_{ZT},	$\bar{1}_{XYZ}$,	$\bar{1}_{XYT}$
04-03-01-001	P	2,	$\bar{1}$,	$\bar{1}$
002		2_b,	$\bar{1}$,	$\bar{1}$
003		2,	$\bar{1}_d$,	$\bar{1}$
004		2_b,	$\bar{1}_d$,	$\bar{1}$
005		2,	$\bar{1}_d$,	$\bar{1}_c$
006		2_b,	$\bar{1}_d$,	$\bar{1}_c$
02-001	$S(Z, T)$	2,	$\bar{1}$,	$\bar{1}$
002		2_b,	$\bar{1}$,	$\bar{1}$
03-001	$S(Y, Z)$	2,	$\bar{1}$,	$\bar{1}$
002		2_a,	$\bar{1}$,	$\bar{1}$
003		2,	$\bar{1}_d$,	$\bar{1}$
004		2_a,	$\bar{1}_d$,	$\bar{1}$
04-001	$I(Y, Z, T)$	2,	$\bar{1}$,	$\bar{1}$
002		2_a,	$\bar{1}$,	$\bar{1}$
003		2,	$\bar{1}$,	$\bar{1}_c$
05-001	$D(X, T)(Y, Z)$	2,	$\bar{1}$,	$\bar{1}$
06-001	$F(Y, Z, T)$	2,	$\bar{1}$,	$\bar{1}$
002		2_a,	$\bar{1}$,	$\bar{1}$

Table 3 continued on p. 554.

Table 3 (*cont.* 2)

IV-4 Orthogonal parallelogram (XY) rectangle (ZT)

		2_{XY},		2_{ZT},	m_Z,	m_T
04-04-01-001	P	2	⊥	2,	m,	m
002		2_d	⊥	2,	m,	m
003		2_{c+d}	⊥	2,	m,	m
004		2_c	⊥	2,	m,	m_c
005		2_{c+d}	⊥	2,	m,	m_c
006		2	⊥	2,	m,	m_c
007		2_d	⊥	2,	m,	m_c
008		2	⊥	2_b,	m,	m_b
009		2_d	⊥	2_b,	m,	m_b
010		2_c	⊥	2_b,	m,	m_b
011		2_{c+d}	⊥	2_b,	m,	m_b
012		2_c	⊥	2_b,	m,	m_{b+c}
013		2_{c+d}	⊥	2_b,	m,	m_{b+c}
014		2	⊥	2_b,	m,	m_{b+c}
015		2_d	⊥	2_b,	m,	m_{b+c}
016		2_{c+d}	⊥	2,	m_d,	m_c
017		2_c	⊥	2,	m_d,	m_c
018		2	⊥	2,	m_d,	m_c
019		2_d	⊥	2_b,	m_d,	m_b
020		2	⊥	2_b,	m_d,	m_b
021		2_{c+d}	⊥	2_b,	m_d,	m_b
022		2_c	⊥	2_b,	m_d,	m_b
023		2_{c+d}	⊥	2_b,	m_d,	m_{b+c}
024		2_c	⊥	2_b,	m_d,	m_{b+c}
025		2_d	⊥	2_b,	m_d,	m_{b+c}
026		2	⊥	2_b,	m_d,	m_{b+c}
027		2	⊥	2,	m_b,	m_b
028		2_d	⊥	2,	m_b,	m_b
029		2_{c+d}	⊥	2,	m_b,	m_b
030		2_c	⊥	2,	m_b,	m_{b+c}
031		2_{c+d}	⊥	2,	m_b,	m_{b+c}
032		2	⊥	2,	m_b,	m_{b+c}
033		2_d	⊥	2,	m_b,	m_{b+c}
034		2	⊥	2_{a+b},	m_b,	m_a
035		2_d	⊥	2_{a+b},	m_b,	m_a
036		2_{c+d}	⊥	2_{a+b},	m_b,	m_a
037		2_c	⊥	2_{a+b},	m_b,	m_{a+c}
038		2_{c+d}	⊥	2_{a+b},	m_b,	m_{a+c}
039		2	⊥	2_{a+b},	m_b,	m_{a+c}
040		2_d	⊥	2_{a+b},	m_b,	m_{a+c}
041		2_{c+d}	⊥	2,	m_{b+d},	m_{b+c}
042		2_c	⊥	2,	m_{b+d},	m_{b+c}
043		2	⊥	2,	m_{b+d},	m_{b+c}
044		2_{c+d}	⊥	2_{a+b},	m_{b+d},	m_{a+c}
045		2_c	⊥	2_{a+b},	m_{b+d},	m_{a+c}
046		2	⊥	2_{a+b},	m_{b+d},	m_{a+c}
02-001	$S(Z,T)$	2	⊥	2,	m,	m
002		2_c	⊥	2,	m,	m_c
003		2	⊥	2_b,	m,	m_b
004		2_c	⊥	2_b,	m,	m_{b+c}
005		2	⊥	2,	m_b,	m_b
006		2_c	⊥	2,	m_b,	m_{b+c}
007		2	⊥	2_{a+b},	m_b,	m_a
008		2_c	⊥	2_{a+b},	m_b,	m_{a+c}

		2_{XY},		2_{ZT},	m_Z,	m_T
04-04-03-001	$S(Y,Z)$	2	⊥	2,	m,	m
002		2_d	⊥	2,	m,	m
003		2	⊥	2_a,	m,	m_a
004		2_d	⊥	2_a,	m,	m_a
005		2	⊥	2_b,	m,	m_b
006		2_d	⊥	2_b,	m,	m_b
007		2_d	⊥	2,	m_d,	m
008		2	⊥	2,	m_d,	m
009		2_d	⊥	2_a,	m_d,	m_a
010		2	⊥	2_a,	m_d,	m_a
011		2_d	⊥	2_b,	m_d,	m_b
012		2	⊥	2_b,	m_d,	m_b
013		2	⊥	2_a,	m_a,	m
014		2_d	⊥	2_a,	m_a,	m
015		2	⊥	2,	m_a,	m_a
016		2_d	⊥	2,	m_a,	m_a
017		2	⊥	2_{a+b},	m_a,	m_b
018		2_d	⊥	2_{a+b},	m_a,	m_b
019		2_d	⊥	2_a,	m_{a+d},	m
020		2	⊥	2_a,	m_{a+d},	m
021		2_d	⊥	2,	m_{a+d},	m_a
022		2	⊥	2,	m_{a+d},	m_a
023		2_d	⊥	2_{a+b},	m_{a+d},	m_b
024		2	⊥	2_{a+b},	m_{a+d},	m_b
04-001	$I(Y,Z,T)$	2	⊥	2,	m,	m
002		2_c	⊥	2,	m,	m
003		2	⊥	2_b,	m,	m_b
004		2_c	⊥	2_b,	m,	m_b
005		2	⊥	2_a,	m,	m_a
006		2_c	⊥	2_a,	m,	m_a
007		2	⊥	2_{a+b},	m_a,	m_b
008		2_c	⊥	2_{a+b},	m_a,	m_b
009		2	⊥	2,	m_a,	m_a
010		2_c	⊥	2,	m_a,	m_a
011		2	⊥	2_b,	m_a,	m_{a+b}
012		2_c	⊥	2_b,	m_a,	m_{a+b}
013		2	⊥	2,	m_b,	m_b
014		2_c	⊥	2,	m_b,	m_b
05-001	$D(X,T)(Y,Z)$	2	⊥	2,	m,	m
002		2	⊥	2_a,	m_a,	m
003		2	⊥	2_{a+b},	m_a,	m_b
06-001	$F(Y,Z,T)$	2	⊥	2,	m,	m
002		2	⊥	2_a,	m,	m_a
003		$2_{(-c+d)/4}$	⊥	2_b,	$m_{(b+d)/4}$,	$m_{(b+c)/4}$
004		$2_{(-c+d)/4}$	⊥	2_{a+b},	$m_{(b+d)/4}$,	$m_{(2a+b+c)/4}$
005		2	⊥	2,	m_a,	m_a

The cell of this system is a parallelotope defined by two parallelograms which belong to the orthogonal planes (XY) and (ZT). A general parallelotope exhibits six sets of four parallel faces: (XY), (XZ), (XT), (YZ), (YT), (ZT).

For the cell considered here, the faces are either parallelograms, such as (XY) and (ZT), or rectangles, such as (XT), (XZ), (YZ) and (YT).

In the space $\mathbb{E}^2$, we recall that only the rectangular cell can be centred, the three others always being primitive (parallelogram, square and hexagon). So there is only one Bravais type S in family III, *e.g.* $S(Y,Z)$, the other two Bravais types being P and $D(X,T)(Y,Z)$, where two sets of rectangles are simultaneously centred.

Then let us consider the orthogonal parallelogram (XY) square (ZT) family (or family VI system 07). The square and one side of the parallelogram generate a right prism based on a square or tetragonal system; in $\mathbb{E}^3$ this system only contains two Bravais types P and I; the same result occurs in $\mathbb{E}^4$. The parallelogram and one side of the square generate a right prism

Table 3 (*cont.* 3)

VI-07 Orthogonal parallelogram (XY) square (ZT)

		$2_{XY}4^1_{ZT}$,
07-01-01-001	P	24
02-001	$I(Y,Z,T)$	24

		4^1_{ZT}
07-02-01-001	P	4
002		$4_{b/4}$
003		4_b
02-001	$I(Y,Z,T)$	4
002		$4_{(-a+b)/4}$
003		4_{a+b}
004		$4_{b/4}$

		2_{XY},		4^1_{ZT}
07-03-01-001	P	2	$\perp$	4
002		2_{c+d}	$\perp$	4
003		2	$\perp$	4_b
004		2_{c+d}	$\perp$	4_b
02-001	$I(Y,Z,T)$	2	$\perp$	4
002		2_c	$\perp$	$4_{b/4}$
003		2	$\perp$	4_{a+b}

		$2_{XY}4^1_{ZT}$,	$\bar{1}_{XYZ}$,	m_{Z-T}
07-04-01-001	P	24,	$\bar{1}$,	m
002		24,	$\bar{1}$,	m_b
003		24,	$\bar{1}_d$,	m
004		24,	$\bar{1}_d$,	m_b
04-001	$I(Y,Z,T)$	24,	$\bar{1}$,	m
002		24,	$\bar{1}$,	$m_{b/4}$
003		24,	$\bar{1}$,	m_{a+b}

		$2_{XY}4^1_{ZT}$,	m_Z,	$\bar{1}_{XYZ-T}$
07-04-02-001	P	24,	m,	$\bar{1}$
002		24,	m_d,	$\bar{1}$
003		24,	m_b,	$\bar{1}$
004		24,	m_{b+d},	$\bar{1}$
03-001	$I(Y,Z,T)$	24,	m,	$\bar{1}$
002		24,	m_{a+b},	$\bar{1}$
003		24,	m_b,	$\bar{1}$

		4^1_{ZT},	$\bar{1}_{XYZ}$,	$\bar{1}_{XYT}$
07-05-01-001	P	4,	$\bar{1}$,	$\bar{1}$
002		$4_{a/4}$,	$\bar{1}$,	$\bar{1}$
003		4_a,	$\bar{1}$,	$\bar{1}$
004		4,	$\bar{1}_d$,	$\bar{1}_c$
005		$4_{a/4}$,	$\bar{1}_d$,	$\bar{1}_c$
006		4_a,	$\bar{1}_d$,	$\bar{1}_c$
02-001	$I(Y,Z,T)$	4,	$\bar{1}$,	$\bar{1}$
002		$4_{(-a+b)/4}$,	$\bar{1}$,	$\bar{1}$
003		4_{a+b},	$\bar{1}$,	$\bar{1}$
004		$4_{b/4}$,	$\bar{1}$,	$\bar{1}$

		4^1_{ZT},	m_Z,	m_T
07-06-01-001	P	4,	m,	m
002		4_a,	m,	m
003		4,	m_d,	m_c
004		4_a,	m_d,	m_c
005		4,	m_b,	m_b
006		4_a,	m_b,	m_b
007		4_b,	m_b,	m_b
008		4,	m_{b+d},	m_{b+c}
009		4_a,	m_{b+d},	m_{b+c}
010		4_b,	m_{b+d},	m_{b+c}
02-001	$I(Y,Z,T)$	4,	m,	m
002		4_a,	m,	m
003		$4_{b/4}$,	m,	m_{b+c}
004		4,	m_{a+b},	m_{a+b}
005		4_a,	m_{a+b},	m_{a+b}
006		$4_{b/4}$,	m_{a+b},	m_{a+c}
007		4,	m_b,	m_b
008		4_a,	m_b,	m_b
009		$4_{b/4}$,	m_b,	m_c

		2_{XY},		4^1_{ZT},	m_Z,	m_T
07-07-01-001	P	2	$\perp$	4,	m,	m
002		2	$\perp$	4,	m_d,	m_c
003		2	$\perp$	4,	m_b,	m_b
004		2	$\perp$	4,	m_{b+d},	m_{b+c}
005		2_{c+d}	$\perp$	4,	m_d,	m_c
006		2_{c+d}	$\perp$	4,	m,	m
007		2_{c+d}	$\perp$	4,	m_{b+d},	m_{b+c}
008		2_{c+d}	$\perp$	4,	m_b,	m_b
009		2	$\perp$	4_b,	m,	m
010		2	$\perp$	4_b,	m_d,	m_c
011		2	$\perp$	4_b,	m_b,	m_b
012		2	$\perp$	4_b,	m_{b+d},	m_{b+c}
013		2	$\perp$	4_b,	m_a,	m_a
014		2	$\perp$	4_b,	m_{a+d},	m_{a+c}
015		2_{c+d}	$\perp$	4_b,	m_d,	m_c
016		2_{c+d}	$\perp$	4_b,	m,	m
017		2_{c+d}	$\perp$	4_b,	m_{b+d},	m_{b+c}
018		2_{c+d}	$\perp$	4_b,	m_b,	m_b
019		2_{c+d}	$\perp$	4_b,	m_{a+d},	m_{a+c}
020		2_{c+d}	$\perp$	4_b,	m_a,	m_a
02-001	$I(Y,Z,T)$	2	$\perp$	4,	m,	m
002		2	$\perp$	4,	m_{a+b},	m_{a+b}
003		2	$\perp$	4,	m_b,	m_b
004		2_c	$\perp$	$4_{b/4}$,	m_{b+d},	m
005		2_d	$\perp$	$4_{b/4}$,	m_{a+d},	m_{a+b}
006		2_d	$\perp$	$4_{b/4}$,	m_d,	m_b
007		2	$\perp$	4_{a+b},	m,	m
008		2	$\perp$	4_{a+b},	m_{a+b},	m_{a+b}
009		2	$\perp$	4_{a+b},	m_b,	m_b

Table 3 continued on p. 556.

based on a parallelogram or monoclinic in $\mathbb{E}^3$ but this point of view concerns the family di orthogonal parallelograms whose centring types are P, S and D.

Lastly the orthogonal parallelogram (XY) rectangle (ZT) family or family IV presents another peculiarity. Only two Bravais types of centred rectangles can appear: either the centred rectangle is orthogonal to the other two vectors of the cell so that it is the Bravais type $S(Z,T)$; or the centred rectangle is not orthogonal to the other two vectors of the cell so that it is the Bravais type $S(Y,Z)$ [or $S(X,Z)$, $S(X,T)$, $S(Y,T)$].

The two types of centring $S(Y,Z)$ and $S(Y,T)$ which appear in the Z classes 04-02-03 and 04-02-04 are in fact one and only one Bravais type; these two Z classes only differ in the position of the axis $\bar{1}$,

Table 3 (*cont.* 4)

VII-08 Orthogonal parallelogram (XY) hexagon (ZT) $R(Y, Z, T)$ centred

		3^1_{ZT}
08-01-01-001	$R(Y, Z, T)$	3
002		$3_{a/3}$
02-001	P	3
002		$3_{b/3}$
		$2_{XY}6^1_{ZT}$
08-02-01-001	$R(Y, Z, T)$	26
02-001	P	26
		3^1_{ZT}, m_Z, 1
08-03-01-001	$R(Y, Z, T)$	3, m, 1
002		3, m_b, 1
02-001	P	3, m, 1
002		3, m_b, 1
		3^1_{ZT}, 1, m_{Z-2T}
08-03-03-001	P	3, 1, m
002		3, 1, m_b

		3^1_{ZT}, $\bar{1}_{XYZ}$, 1
08-04-01-001	$R(Y, Z, T)$	3, $\bar{1}$, 1
002		$3_{a/3}$, $\bar{1}$, 1
02-001	P	3, $\bar{1}$, 1
002		$3_{b/3}$, $\bar{1}$, 1
		3^1_{ZT}, 1, $\bar{1}_{XYZ-2T}$
08-04-03-001	P	3, 1, $\bar{1}$
002		$3_{b/3}$, 1, $\bar{1}$
		$2_{XY}6^1_{ZT}$, m_Z, $\bar{1}_{XYZ+T}$
08-05-01-001	$R(Y, Z, T)$	26, m, $\bar{1}$
002		26, m_b, $\bar{1}$
02-001	P	26, m, $\bar{1}$
002		26, m_b, $\bar{1}$
		$2_{XY}6^1_{ZT}$, $\bar{1}_{XYT}$, m_{Z+T}
08-05-03-001	P	26, $\bar{1}$, m
002		26, $\bar{1}$, m_b

VII-09 Orthogonal parallelogram (XY) hexagon (ZT)

		6^1_{ZT}
09-01-01-001	P	6
002		$6_{b/6}$
003		$6_{b/3}$
004		6_b
		2_{XY}, 3^1_{ZT}
09-02-01-001	P	$2 \perp 3$
		2_{XY}, 6^1_{ZT}
09-03-01-001	P	$2 \perp 6$
002		$2 \perp 6_b$
		6^1_{ZT}, m_Z, m_T
09-04-01-001	P	6, m, m
002		6_b, m, m
003		6, m_b, m_b
004		6_b, m_b, m_b
005		6_a, m_b, m_b

		6^1_{ZT}, $\bar{1}_{XYZ}$, $\bar{1}_{XYT}$
09-05-01-001	P	6, $\bar{1}$, $\bar{1}$
002		$6_{b/6}$, $\bar{1}$, $\bar{1}$
003		$6_{b/3}$, $\bar{1}$, $\bar{1}$
004		6_b, $\bar{1}$, $\bar{1}$
		2_{XY}, 3^1_{ZT}, m_Z, 1
09-06-01-001	P	$2 \perp 3$, m, 1
002		$2 \perp 3$, m_b, 1
		2_{XY}, 3^1_{ZT}, 1, m_{Z+T}
09-06-02-001	P	$2 \perp 3$, 1, m
002		$2 \perp 3$, 1, m_b
		2_{XY}, 6^1_{ZT}, m_Z, m_T
09-07-01-001	P	$2 \perp 6$, m, m
002		$2 \perp 6_b$, m, m
003		$2 \perp 6$, m_b, m_b
004		$2 \perp 6_b$, m_b, m_b
005		$2 \perp 6_a$, m_b, m_b

which is either parallel to a side of the rectangle or parallel to a side of the parallelogram. These two Z classes would be denoted $2, m, \bar{1}$ $S(Y, Z)$ and $2, \bar{1}, m$ $S(Y, Z)$ instead of $2/m$ but in order to avoid changing the symbol of the point group of this Q class, we prefer the introduction of another type of centring: $S(Y, Z)$ for the Z class 04-02-03 and $S(Y, T)$ for the Z class 04-02-04, and we add the PSO $\bar{1}$ in the symbol of the point group which is therefore denoted as $2/m, \bar{1}$.

Thus we can write the supports of the translation vectors in a precise way for all the space groups of this Z class.

Table 4. *Di orthogonal parallelograms* (XY) (ZT) *system*

In the first column we give the names of the three Bravais types of this system. The letters P, S and D are defined in the caption to Table 2. The second column contains the three corresponding Bravais flocks.

Bravais types of lattices	Bravais flocks
Di orthogonal parallelograms P	$P2$; $P2_d$; $P2 \perp 2$; $P2_d \perp 2$; $P2_d \perp 2_b$
Di orthogonal parallelograms $S(Y, Z)$	$S(Y, Z)2$; $S(Y, Z)2_d$; $S(Y, Z)\ 2 \perp 2$; $S(Y, Z)2_d \perp 2$; $S(Y, Z)2_d \perp 2_a$
Di orthogonal parallelograms $D(X, T)(Y, Z)$	$D(X, T)(Y, Z)2$; $D(X, T)(Y, Z)2 \perp 2$

Table 5. *WPV notation of the* 76 *crystallographic Z classes of the seven mono-incommensurate systems*

We give the WPV notation of the 76 Z classes classified by family. The centring types P, S, D, F, I and R are explained in the caption to Table 2. In family IV 2, m, $\bar{1}$ and 2, $\bar{1}$, m are more convenient symbols for the Z classes than the symbol $2/m$ which is that of the corresponding Q class. Nevertheless we keep the WPV symbol $2/m$ for the polar point group.

Family I	Family IV	Family VI	Family VII
$1\,P$	$2, m, m\,P$	$24\,P$	$3\,R$
$\bar{1}_4\,P$	$2, m, m\,S(Z, T)$	$24\,I$	$3\,P$
	$2, m, m\,S(Y, Z)$	$4\,P$	$26\,R$
	$2, m, m\,I$	$4\,I$	$26\,P$
Family II	$2, m, m\,D$	$2\perp 4\,P$	$3, m\,R$
$m\,P$	$2, m, m\,F$	$2\perp 4\,I$	$3, m, 1\,P$
$m\,S$	$2, m, \bar{1}\,P$	$24, \bar{1}, m\,P$	$3, 1, m\,P$
$\bar{1}\,P$	$2, m, \bar{1}\,S(Z, T)$	$24, \bar{1}, m\,I$	$3, \bar{1}\,R$
$\bar{1}\,S$	$2, m, \bar{1}\,S(Y, Z)$	$24, m, \bar{1}\,P$	$3, \bar{1}, 1\,P$
$\bar{1}\perp m\,P$	$2, \bar{1}, m\,S(Y, Z)$	$24, m, \bar{1}\,I$	$3, 1, \bar{1}\,P$
$\bar{1}\perp m\,S$	$2, m, \bar{1}\,I$	$4, \bar{1}, \bar{1}\,P$	$26, m, \bar{1}\,R$
	$2, m, \bar{1}\,D$	$4, \bar{1}, \bar{1}\,I$	$26, m, \bar{1}\,P$
	$2, m, \bar{1}\,F$	$4, m, m\,P$	$26, \bar{1}, m\,P$
Family III	$2, \bar{1}, \bar{1}\,P$	$4, m, m\,I$	
$2\,P$	$2, \bar{1}, \bar{1}\,S(Z, T)$	$2\perp 4, m, m\,P$	$6\,P$
$2\,S$	$2, \bar{1}, \bar{1}\,S(Y, Z)$	$2\perp 4, m, m\,I$	$2\perp 3\,P$
$2\,D$	$2, \bar{1}, \bar{1}\,I$		$2\perp 6\,P$
$2\perp 2\,P$	$2, \bar{1}, \bar{1}\,D$		$6, m, m\,P$
$2\perp 2\,S$	$2, \bar{1}, \bar{1}\,F$		$6, \bar{1}, \bar{1}\,P$
$2\perp 2\,D$	$2\perp 2, m, m\,P$		$2\perp 3, m, 1\,P$
	$2\perp 2, m, m\,S(Z, T)$		$2\perp 3, 1, m\,P$
	$2\perp 2, m, m\,S(Y, Z)$		$2\perp 6, m, m\,P$
	$2\perp 2, m, m\,I$		
	$2\perp 2, m, m\,D$		
	$2\perp 2, m, m\,F$		

Finally, in Table 5, we give a list of the 76 MI Z classes in $\mathbb{E}^4$ which corresponds to the same subdivisions as those of Table 3.

III. Concluding remarks

We can compare MI crystallographic symmetries in the superspace $\mathbb{E}^4$ to the crystallographic symmetries in the physical space $\mathbb{E}^3$, and, in particular, the number of space groups and Z classes.

	MI crystal in $\mathbb{E}^4$	Crystal in $\mathbb{E}^3$
Space groups (Table 3)	371	219
Z classes (Table 5)	76	73
Point groups	30	32
Bravais types (Table 2)	16	14
Crystal systems	7	7
Crystal families	6	6.

If we allocate a particular role to the direction of the physical modulation, one of the 30 point groups, the point group 2, splits into two point groups 2 and $2^{\blacktriangledown}$ as we saw in a previous paper (Veysseyre *et al.*, 1989).

Lastly, we state the correlation between five of these six crystal families.

MI crystal families in $\mathbb{E}^4$	Crystal families in $\mathbb{E}^3$
Right hyperprism based on triclinic	Triclinic
Di orthogonal parallelograms	Monoclinic
Orthogonal parallelogram rectangle	Orthorhombic
Orthogonal parallelogram square	Tetragonal
Orthogonal parallelogram hexagon	Hexagonal
We note that the last family splits into two systems:	
Orthogonal parallelogram equilateral triangle	Rhombic
Orthogonal parallelogram hexagon	Hexagonal.

In a later paper a correspondence will be established between the notation proposed here and the notation given by de Wolff, Janssen & Janner (1981).

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